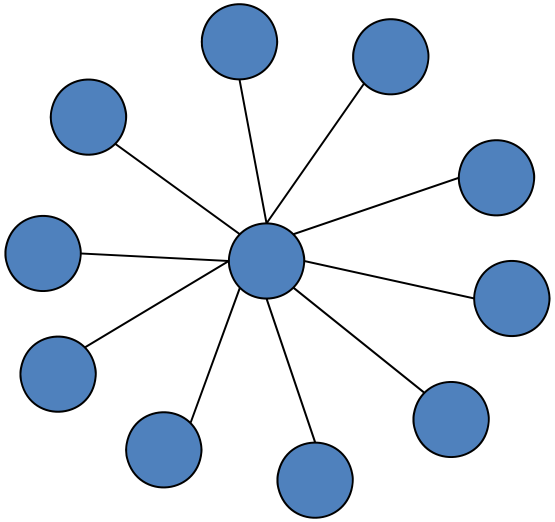
Games on Networks



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Introduction

Individuals are located on nodes of a network. They choose actions and their rewards depend on these actions along with the actions of others on the network. The key point:

- The effect of player 1's action on player 2's payoff depends on where the two players are located in a network.

Examples: 1. value of learning a language depends on how many friends and colleagues learn the same language. 2. Value of acquiring information on market prices depend on how much information friends acquire. 3. Firms collaborate but also compete in markets.

- There are two basic building blocks: one, the formal description of the pattern of relationships among individual entities and two, a description of the externalities that an individual's actions create for other individuals and how these are mediated by the pattern of ties between them. We ask:
- What are the effects of network location on individual behavior and well being? For instance, do better connected individuals earn larger payoffs as compared to poorly connected individuals?
- How does individual behavior and well being respond to changes – the adding of links or the redistribution of links – in a network?

  Are some networks better for the attainment of socially desirable outcomes?

- Two conceptual issues:
  1. A single action or link specific actions? Single action is reasonable is some contexts: such as consumer search about product prices. While in other contexts link specific action is more natural – e.g., effort in research projects, individuals or firms have the choice of putting different amounts of resources in different projects. The single action formulation is simpler and most applications to date assume this.
  2. Direct or indirect effects of actions? Both will be studied.

Games on fixed networks

Suppose each player i takes an action s_i in S, where X is a compact subset of $[0, 1]$. The payoff (utility or reward) to player i under the profile of actions $s = (s_1, ..., s_n)$ is given by
$\Pi_i : S^n \times \mathcal{G} \to \mathcal{R}$.
In the games where the action set S is continuous, it will be assumed that S is also convex. Define s_{-i} as the profile of all strategies other than player i.

- *Networks in the payoff function:* A network has a number of different attributes – such as neighborhood size, average path length, degree distribution, centrality – and it is clear that these attributes will play more or less important roles depending on the particular context under study.
- *Neighbors and non-neighbors*. The simplest way to model this is to classify other players into two categories, *neighbors* and *non-neighbors*, and to treat all members in each group alike. The effects of actions of neighbors are then termed *local* effects, while the actions of non-neighbors are termed *global* effects.

Pure local effects: Only neighbors matter Define the function $\phi_k : S^k \to \mathcal{R}$. In this case:

$$\Pi_i(s|g) = \phi_{\eta_i(g)}(s_i, \{s_j\}_{j \in N_i(g)})$$

Remarks: same payoffs of players with same degree; Two, the payoff function is anonymous. If $\{s_j'\}_{j \in N_i(g)}$ is a permutation of actions in $\{s_j\}_{j \in N_i(g)}$ then

$$\phi_{\eta_i(g)}(s_i, \{s_j'\}_{j \in N_i(g)}) = \phi_{\eta_i(g)}(s_i, \{s_j\}_{j \in N_i(g)}).$$

Pure global effects: the actions of all players have the same effects on payoffs.

$$\Pi_i(s|g) = \phi_{n-1}(s_i, s_{-i}).$$

A combined model of local and global effects: Define a function $f_k : S^k \to R$, where k is the degree of a player i, and define a function $h_k : S^{n-k-1} \to \mathcal{R}$. Assume that functions respect anonymity of actions and assume that they are same across players.

$$\Pi_i(s|g) = \Phi(s_i, f_{k_i}(\{s_j\}_{j \in N_i(g)}), h_k(s)). \tag{1}$$

An important special case of this framework arises when the payoff depends simply on the sum of neighbors actions and the sum of non-neighbors actions.

$$\Pi_i(s|g) = \Phi(s_i, \sum_{j \in N_i(g)} s_j, \sum_{k \notin N_i(g) \cup \{i\}} s_k). \tag{2}$$

Nature of externality: Effects of others actions on payoffs as well as marginal payoffs. In the polar cases, pure local effects or pure global effects, it is easy to define them.

Definition

A game with pure local effects satisfies positive externality if for each ϕ_k, and for $s, s' \in S^k$, if $s \geq s'$ then $\phi_k(s_i, s) \geq \phi_k(s_i, s')$. Similarly, the game exhibits *negative externality* if for each ϕ_k, and for $s, s' \in S^k$, if $s \geq s'$ then $\phi_k(s_i, s) \leq \phi_k(s_i, s')$.

Definition

A game with pure local effects exhibits strategic complements (substitutes), if for all ϕ_k, $s_i > s_i'$, $s, s' \in S^k$, if $s_i \geq s_i'$ then $\phi_k(s_i, s) - \phi_k(s_i', s) \geq (\leq) \phi_k(s_i, s') - \phi_k(s_i', s')$.

Example 1: Bramoulle and Kranton 2007. There are n players. Suppose each player chooses a search intensity $s_i \in S$, where S is a compact and convex interval in $\mathcal{R}_+$. The payoffs to a player i, in a network g, faced with a profile of efforts $s = \{s_1, s_2, ..., s_n\}$, are given by:

$$\Pi_i(s|g) = f(s_i + \sum_{j \in N_i(g)} s_j) - cs_i \tag{3}$$

where $c > 0$ is the marginal cost of effort. It is assumed that $f(0) = 0$, $f' > 0$ and $f'' < 0$.
A game of pure local effects. It is also a game of positive externality and strategic substitutes.

Example 2: Ballester, Calvo-Armengol, and Zenou (2006)
The role of interaction effects in shaping the level of criminal activity has been a recurring theme is different literatures such as social psychology and economics. Consider a n player game with linear quadratic payoffs. The payoffs to player i faced under strategy profile s, are given by:

$$\Pi_i(s) = \alpha.s_i + \frac{1}{2}\rho s_i^2 + \sum_{j \neq i} \gamma g_{ij} s_i.s_j \tag{4}$$

Assume that $\alpha > 0$, $\rho < 0$ and $\gamma > 0$. So this is a game of pure local effects, positive externalities and complements.

Example 3: Goyal and Moraga 2001. Firms increasingly choose to collaborate in research with other firms. This research collaboration takes a variety of forms and is aimed both at lowering costs of production as well as improving product quality and introducing entirely new products.
Suppose demand is linear and given by $Q = 1 - p$ and that the initial marginal cost of production in a firm is $\bar{c}$ and assume that $n\bar{c} < 1$. Each firm i chooses a level of research effort given by $s_i \in S = [0, \bar{c}]$. Collaboration between firms is at a bilateral level and it allows for firms to share research efforts which lower costs of production. The marginal costs of production of a firm i, in a network g, facing a profile of efforts s, are given by:

$$c_i(s|g) = \bar{c} - (s_i + \sum_{j \in N_i(g)} s_j). \tag{5}$$

The cost of efforts is given by $Z(s_i) = \alpha s_i^2/2$, where $\alpha > 0$. Given costs $c = \{c_1, c_2, ...c_n\}$, firms choose quantities $(\{q_i\}_{i \in N})$, with $Q = \sum_{i \in N} q_i$. Solve for market quantity equilibrium given any cost profile. Then payoffs of firm i, located in network g, faced with a research profile s is:

$$\left[\frac{1 - \bar{c} + s_i[n - \eta_i] + \sum_{j \in N_i(g)} s_j[n - \eta_j(g)] - \sum_{l \in N \setminus \{i\} \cup N_i} s_l[1 + \eta_l(g)]}{n+1}\right]$$

This game exhibits local & global effects. Positive externality across neighbors and negative externality across non-neighbors actions. Actions of neighbors are complements, while the actions of non-neighbors are substitutes.

Local public goods

- Existence of Nash equilibrium: The action set is compact, the payoffs are continuous in actions of all players are concave in own action, it follows from standard theorem that there is Nash equilibrium in pure strategies.
- Networks and equilibrium: A useful first step is a general property concerning aggregate level of effort – own plus the neighborhood – enjoyed by any individual. Let $\hat{s}$ be such that $f'(\hat{s}) = c$ and define $\bar{s}_{N_i(g)} = \sum_{j \in N_i} s_j$. From the concavity of $f(.)$, it then follows that if $\bar{s}_{N_i(g)} \geq \hat{s}$ then marginal returns to effort are lower than the marginal cost and so optimal effort is 0, while if $\bar{s}_{N_i(g)} < \hat{s}$, then marginal returns from effort to player are strictly larger than marginal costs c and so optimal effort is positive and in fact given by $\hat{s} - \bar{s}_{N_i(g)}$.

Proposition

A profile of actions $s^* = \{s_1, s_2, ..., s_n\}$ is a Nash equilibrium if and only if for every player i either (1). $\bar{s}^*_{N_i(g)} \geq \hat{s}$ or (2). $\bar{s}^*_i \leq \hat{s}_i$ and $s^*_i = \hat{s} - \bar{s}^*_{N_i(g)}$.

There are two types of players: those who receive aggregate effort in excess of $\hat{s}$ from their neighbors and exert no effort on their own, and two, players who receive less than $\hat{s}$ aggregate effort from their neighbors and contribute exactly the difference between what they receive and $\hat{s}$.

Specialized equilibria: profile where some players choose positive action while others choose 0 action. such profiles illustrate free riding in a specially acute form.

Point 1: In the empty network – there is a unique equilibrium in which every player chooses $\hat{s}$: no free riding. It turns out that this is the only network in which no free riding is possible. How do we prove this?

- An *independent set* of a network g is a set of players $I \subseteq N$ such that for any $i, j \in I$, $g_{ij} \neq 1$.
- A *maximal independent set* is an independent set that is not contained in any other independent set. Does there exist a maximal independent set in every network? The answer to this is yes. Proof by construction:
- First number the players $1, 2, ...n$. start by placing player 1 in I. If player $2 \notin N_i(g)$, then include her in the independent set,I, if not then include her in the complement set I^c. Next consider player 3: if player $3 \notin N_1(g) \cup N_2(g)$, then include her in I, while if she is not then include her in I^c. Move next to player 4 and proceed likewise until you reach player n.

- Examples: In the empty network there exists a unique maximal independent set and this is the set of all players N. In the complete network on the other hand, there are n distinct maximal sets, each of which contain a single player. In the star network, there are two maximal independent sets, one, which contains the central player, and two, which contains all the peripheral players.
- Now assign the action $\hat{s}$ to every member of a maximal independent set and assign action 0 to every player who is not a member of the maximal independent set. This configuration constitutes an equilibrium in view of the characterization provided in Proposition 1. Such an equilibrium is by construction a non-trivial *specialized* equilibrium.

In any non-empty network, a maximal independent set must be a *strict* subset of the set of players N.

Proposition

There exists a specialized equilibrium in every network. In the empty network the unique equilibrium is specialized and every player chooses $\hat{s}$*, so there is no free riding. In any non-empty network there exists a specialized equilibrium with free riding.*

Network advantages

- Specialized equilibria point to unequal effort. Does this translate into unequal payoffs?
- Network location is a rather general idea and there are different aspects of networks that may play a role. Here with positive payoff externality, the intuition is that higher degree players should earn more. Is this true?
- **Example:** Star network. The two equilibria are both specialized and have clearly unequal payoffs.
- It is difficult to say anything definite on relation between degree and payoff. knowledge.

Network structure and social welfare: Aggregate welfare from a strategy profile s in network g is defined as:

$$W(s|g) = \sum_{i \in N} [f(s_i + s_{N_i(g)} - cs_i].$$

Given a network g, a strategy profile profile s is efficient if there is no other action profile s' such that $W(s'|g) > W(s|g)$.

Proposition

Every equilibrium in a non-empty network is inefficient.

This result is a direct consequence of individual actions having positive externality on others.

Effects of adding links: example

- Start with two stars each with 3 peripheral players. Fix an equilibrium in which the two centers exert action $\hat{s}$ while the peripheral players all choose 0.
- *First* add a link between a center and a spoke of the other star. The old action profile still constitutes an equilibrium. It then follows that aggregate welfare increases on the adding of a link.
- *Second*, add a link between the centers. The two centers do not constitute a maximal independent set any more. Equilibrium must change. The best equilibrium is one in which the center of one star and spokes of the other star choose $\hat{s}$, and all other players choose 0. In this new equilibrium, welfare strictly decreases if $2c\hat{s} > f(4\hat{s}) - f(\hat{s})$.
- Thus effects of adding links depend on subtle details of the network.

General observations

- Important role of networks in sustaining specialized equilibria: creating significant free riding and payoff inequality.
- Connections and network advantages: Multiple equilibria exist, with contrasting relationship.
- Adding links: increases payoffs in some standard networks but not for other simple networks!
- A key limitation is the multiplicity of equilibria, with very different properties.
- Follow up work see Amours, Bramoulle and Kranton (2011), Strategic Interaction and Networks.

4. Games on random networks

- Introspection and casual observation both suggest that individuals have very incomplete information about the network: know their own connections and have information about some statistics of the network.
- Aim: present a framework to study behavior in networks when players have incomplete information about the network.

Main findings:

1. Location in network: behavior and payoffs are monotonic in degree: increasing in games with complements, decreasing in games with substitutes.
2. Payoffs increasing (decreasing) in degree if game has positive (negative) externality.
3. Adding links: model in terms of dominance relations of degree distributions. Show how effects depend on whether game exhibits complements or substitutes.

Basic notation:

- *Players:* $\mathcal{N} = \{1, 2, ..., n\}$ Each player i is a node i of an undirected network g; $g_{i,j} = 1, 0$ indicates presence/absence of link between i and j,, respectively.
- *Neighbors:* $\mathcal{N}_i(g) = \{j \in \mathcal{N} : g_{i,j} = 1\}$, is set of neighbors and $\eta_i(g) = |\mathcal{N}_i(g)|$ is her degree in g.
- *Actions:* Player i chooses $x_i \in \mathcal{X}$, where $\mathcal{X}$ is a compact subset of $[0, 1]$. We allow for both continuous and discrete actions sets.

- **Payoffs:** The payoff of player i under $x = (x_1, \ldots, x_n)$ is:

$$u_i(x, g) = v_{k_i(g)}(x_i, x_{N_i(g)})$$

where $x_{N_i(g)}$ is the vector of actions taken by the neighbors of i.

– note anonymity of neighbors and homogeneity across players with same degree.

- Assume that for any x_i and k-dimensional vector x:

$$v_{k+1}(x_i, (x, 0)) = v_k(x_i, x). \qquad (A)$$

- Thus adding a link to a neighbor who chooses action 0 is similar to not having an additional neighbor. Rules out payoffs, e.g., product or average of neighbors actions.

- A game exhibits strict *strategic complements* if it satisfies increasing differences. That is, for all k, $x_i > x_i'$, and $x > x'$: $v_k(x_i, x) - v_k(x_i', x) > v_k(x_i, x') - v_k(x_i', x')$. Analogously for substitutes.
- A game exhibits *positive externalities* if for each v_k, and for all $x \geq x'$, $v_k(x_i, x) \geq v_k(x_i, x')$. Negative externalities analogously defined.

- An example where payoffs depend on sum of actions.

$$v_k\left(x_i, \sum_{j=1}^{k} x_j\right) = f(x_i + \lambda \sum_{j=1}^{k} x_j) - c(x_i) \quad (6)$$

  where $c(\cdot)$ is the cost of action and $f(\cdot)$ is the gross return. Clearly satisfies Assumption A.
- Bramoulle and Kranton (2007): public goods model if $\lambda = 1$, $f(\cdot)$ is concave, $c(\cdot)$ is linear and increasing. [(strict) strategic substitutes and positive externalities.]
- Goyal and Moraga (2001): collaboration among monopolies if $\lambda = 1$, $f''(\cdot) > 0$ and $c''(\cdot) > f''(\cdot) > 0$. [(strict) strategic complements and positive externalities].

- Local information is reflected in the knowledge of own degree, while information concerning the network at large is reflected in knowledge of the aggregate distribution of degrees across the society.
- Let $P(\cdot)$ be the unconditional probability that any given node has a given degree $P(k_i)$.
- Let the degrees of the neighbors of a player i of degree k_i be denoted by $\mathbf{k}_{N(i)}$, which is a vector of dimension k_i.
- **Assume that degrees of neighbors are independent, as in the classical Erdos-Renyi model of random networks (in the limit, when n is infinite).**

- The strategy for player i is a mapping $\sigma_i : \{0, 1, ..., n-1\} \to \Delta(X)$, where $\Delta(X)$ is the set of distribution functions on X. We say σ is *nondecreasing* if $\sigma(k')$ first order stochastic dominates $\sigma(k)$ for each $k' > k$. Similarly, for *nonincreasing*. Given a player i of degree k_i let $\mathrm{d}\psi_{-i}(\sigma, k_i)$ denote the probability measure over $x_{N_i} \in X^{k_i}$ induced by the beliefs $P(\cdot \mid k_i)$ composed with the strategies played via σ. The expected payoff to a player is given by:

$$U(x_i, \sigma, k_i) = \int_{x_{N(i)} \in X^{k_i}} v_{k_i}(x_i, x_{N(i)}) \mathrm{d}\psi_{-i}(\sigma, k_i)$$

- An equilibrium is a (Bayesian) Nash equilibrium of this game, in the standard fashion.

Proposition

There exists a symmetric Bayes-Nash equilibrium in the game.

This follows from standard results so long as types of players are finite and strategy sets are either finite or compact convex sets and payoffs are continuous in all actions and concave in own action (if action sets are compact convex sets).

Proposition

Suppose assumption A is satisfied and degrees of neighbors are independent. Every symmetric equilibrium is monotone increasing (decreasing) if payoffs satisfy the strict strategic complements (substitutes) property.

- Intuition: Consider strategic complements. Given assumption A, the best response of a k+1 degree player would be the same as a degree k player if the $k+1$'th player chooses 0, *for sure.* However, in a non-trivial equilibrium, the $(k+1)$'th neighbor would be choosing, on average, a positive action. Strict complementarities imply that the $k+1$ degree player best responds with strictly higher actions than her k degree peers.

Proof: We present the proof for the case of strategic complements. The proof for the case of strategic substitutes is analogous and omitted. Let $\{\sigma_k^*\}$ be the strategy played in a symmetric equilibrium of the network game. If $\{\sigma_k^*\}$ is a trivial strategy with all degrees choosing action 0 with probability 1, the claim follows directly. Therefore, from now on, we assume that the equilibrium strategy is non-trivial and that there is some k' and some $x' > 0$ such that $x' \in \mathbf{supp}(\sigma_{k'}^*)$.

Consider any $k \in \{0, 1, ..., n\}$ and let $x_k = \sup[\mathbf{supp}(\sigma_k^*)]$. If $x_k = 0$, it trivially follows that $x_{k'} \geq x_k$ for all $x_{k'} \in \mathbf{supp}(\sigma_{k'}^*)$ with $k' > k$. So let us assume that $x_k > 0$. Then, for any $x < x_k$, Property A and the assumption of (strict) strategic complements imply that

$$v_{k+1}(x_k, x_{l_1}, ., x_{l_k}, x_s) - v_{k+1}(x, x_{l_1}, ., x_{l_k}, x_s) \geq v_k(x_k, x_{l_1}, ., x_{l_k}) - v_k(x, x_{l_1}$$

for any x_s, with the inequality being strict if $x_s > 0$.

Then, taking expectations across types, noting that degrees of any two neighboring nodes are independent, and that there are some players with degree k who choose $x_k > 0$ implies that

$$U(x_k, \sigma^*, k+1) - U(x, \sigma^*, k+1) > U(x_k, \sigma^*, k) - U(x, \sigma^*, k).$$

On the other hand, note that from the choice of x_k,

$$U(x_k, \sigma^*, k) - U(x, \sigma^*, k) \geq 0$$

for all x. Combining the aforementioned considerations we conclude:

$$U(x_k, \sigma^*, k+1) - U(x, \sigma^*, k+1) > 0,$$

for all $x < x_k$. This means that if $x \in \mathbf{supp}(\sigma^*_{k+1})$ then $x \geq x_k$, which of course implies that σ^*_{k+1} FOSD σ^*_k. Iterating the argument as needed, the desired conclusion follows, i.e., $\sigma^*_{k'}$ FOSD σ^*_k whenever $k' > k$. ■

Proposition

Suppose that payoffs satisfy Assumption A and they are either strict strategic substitutes or complements. Then under positive externalities the expected payoffs are non-decreasing in degree, while under negative externalities the expected payoffs are non-increasing in degree.

- Intuition: Positive externality: suppose that k neighbors of $k+1$ degree player follow the equilibrium strategy, but her $(k+1)'^{th}$ neighbor chooses 0. Assumption A implies that she can earn payoffs as high as a k degree player by imitating this player.
- The payoff degree relation depends only on externality and is independent of complements or substitutes.

Proof: We present the proof for positive externalities. The proof for negative externalities is analogous and omitted. The claim is obviously true for a trivial equilibrium in which all players choose the action 0 with probability 1. So, let σ^* be a (non-trivial) equilibrium strategy. Suppose $x_k \in \textbf{supp}(\sigma_k^*)$ and $x_{k+1} \in \textbf{supp}\left(\sigma_{k+1}^*\right)$. Property A implies that

$$v_{k+1}(x_k, x_{l_1}, ..., x_{l_k}, 0) = v_k(x_k, x_{l_1}, ..., x_{l_k}),$$

for all $x_{l_1}, ..., x_{l_k}$. Since the payoff structure satisfies positive externalities, it follows that for any $x > 0$,

$$v_{k+1}(x_k, x_{l_1}, ..., x_{l_k}, x) \geq v_k(x_k, x_{l_1}, ..., x_{l_k}).$$

Looking at expected utilities, we obtain that:

$$U(x_k, \sigma^*, k+1) \geq U(x_k, \sigma^*, k).$$

Since σ^*_{k+1} is a best response in the network game being played and $x_{k+1} \in \mathbf{supp}\left(\sigma^*_{k+1}\right)$,

$$U(x_{k+1}, \sigma^*, k+1) \geq U(x_k, \sigma^*, k+1)$$

and the result follows.

Local public goods.. again

- In the Bramoulle and Kranton (2007) game of local public goods, payoffs exhibit substitutes and positive externalities. So we apply propositions 5 and 6 to obtain:
- More connected players choose lower effort and hence provide less public goods.
- More connected players earn higher payoffs, thus network connections render clear payoff benefits.
- This is in marked contrast to the original complete information (regarding network) setting, where both actions and payoffs bear no systematic relation to networks. Thus incomplete network knowledge is more plausible and yields sharper equilibrium predictions!

Readings

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